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A BERNSTEIN RESULT FOR MINIMAL GRAPHS OF CONTROLLED GROWTH

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It is well known that the only entire solutions of the minimal surface equation on $\mathbf{R}^n$,

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)=0,$$

are affine functions, provided that either $n \le 7$, [1], [2], [7], [8], [14] or u grows at most linearly [4], [6], [11].

Recently Caffarelli, Nirenberg and Spruck [5] extended this theorem to the case where it is merely assumed that

$$|Du(x)| = o(|x|^{1/2}).$$

Their result was in fact obtained for a general class of nonlinear elliptic equations.

Using the stong geometric information contained in the Codazzi equations we establish the following theorem for minimal surfaces.

Theorem. *An entire smooth solution u of the minimal surface equation satisfying*

$$|Du(x)| = o(\sqrt{|x|^2+|u(x)|^2})$$

is an affine function.

Our result follows from the curvature estimate

$$|A|v(0) \le c(n)R^{-1} \sup_{M\cap B_R} v \tag{1}$$

for $M = \operatorname{graph} u$, which holds for arbitrary entire minimal graphs. Here $0 \in M$, $M \cap B_R = \{(x, u(x)) \in \mathbf{R}^{n+1} | |x|^2 + |u(x)|^2 \le R^2\}$, $v = \sqrt{1+|Du|^2}$ and $|A|$ denotes the norm of the second fundamental form of M.

Notice that (1) still implies a global bound on $|A|v$ in case v grows linearly.

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For an account of nonlinear minimal graphs we refer to [3], [13]. The example in [3] behaves like

$$|u(x)| = O(|x|^{2+O(1/n)}),$$

and [13, Chapter 6] contains nontrivial minimal graphs the gradient of which satisfies

$$|Du(x)| \leq c|x|^{1+O(1/n)}.$$

As for some of these examples $|A(x)| \sim |x|^{-1}$, estimate (1) is optimal.

To prove (1) we recall two well-known relations for minimal surfaces, the Jacobi field equation for minimal graphs

$$\Delta v = |A|^2 v + 2v^{-1}|\nabla v|^2 \tag{2}$$

and Simons' identity [14]

$$\Delta|A|^2 = -2|A|^4 + 2|\nabla A|^2, \tag{3}$$

where ∇ and Δ denote covariant differentiation and the Laplace-Beltrami operator on M respectively.

As was shown in [12] the Codazzi equations imply the inequality

$$\Delta|A|^2 \geq -2|A|^4 + 2(1 + 2/n)|\nabla|A||^2. \tag{4}$$

From (2) and (4) we compute

$$\begin{aligned}\Delta(|A|^p v^q) \geq (q-p)|A|^{p+2}v^q &+ p(p-1+2/n)|A|^{p-2}v^q|\nabla|A||^2 \\ &+ q(q+1)v^{q-2}|A|^p|\nabla v|^2 \\ &+ 2pq|A|^{p-1}v^{q-1}\nabla|A| \cdot \nabla v.\end{aligned}$$

Using Young's inequality we derive

$$\Delta(|A|^p v^q) \geq (q-p)|A|^{p+2}v^q \tag{5}$$

for $p \geq 2$ and $q(1-2/n) \leq p - 1 + 2/n$. In particular for $q = p \geq (n-2)/2$ we obtain

$$\Delta(|A|^p v^p) \geq 0. \tag{6}$$

A standard mean value inequality on minimal surfaces [9, Chapter 16] can be applied to yield

$$|A|^p v^p(0) \leq c(n)R^{-n/2}\left(\int_{M\cap B_R} |A|^{2p}v^{2p}\, d\mathscr{H}^n\right)^{1/2}, \tag{7}$$

where we used the fact that the n-dimensional Hausdorff measure on minimal graphs can be estimated by $\mathscr{H}^n(M \cap B_R) \leq c(n)R^n$ [9, Chapter 16].

In order to estimate the integral on the right-hand side of (7) we derive from (5) for $p \geq \max(3, n-1)$ fixed

$$\Delta(|A|^{p-1}v^p) \geq |A|^{p+1}v^p. \tag{8}$$

We then multiply (8) by $|A|^{p-1}v^p\eta^{2p}$ where η is a test function with compact support. Integrating by parts in conjunction with Young's inequality leads to

$$\int_M |A|^{2p}v^{2p}\eta^{2p}d\mathscr{H}^n \leq c(p)\int_M |A|^{2(p-1)}v^{2p}\eta^{2(p-1)}|\nabla\eta|^2\,d\mathscr{H}^n.$$

In view of the inequality

$$ab \leq \varepsilon\left(\frac{p-1}{p}\right)a^{p/(p-1)} + \frac{\varepsilon^{-(p-1)}}{p}b^p$$

we finally arrive at

$$\int_M |A|^{2p}v^{2p}\eta^{2p}\,d\mathscr{H}^n \leq c(p)\int_M v^{2p}|\nabla\eta|^{2p}\,d\mathscr{H}^n. \tag{9}$$

We now choose η to be the standard cut-off function for $M \cap B_R$. Then, since $p = p(n)$, we obtain from (9)

$$\left(\int_{M\cap B_R} |A|^{2p}v^{2p}\,d\mathscr{H}^n\right)^{1/2} \leq c(n)R^{n/2}R^{-p}\sup_{M\cap B_{2R}} v^p \tag{10}$$

which in view of (7) implies estimate (1).

Note added in proof. The authors were recently informed by J. C. C. Nitsche that in the case $|Du(x)| = O(|x|^\mu)$, $\mu < 1$, a proof of the corresponding result was obtained in his book *Lectures on minimal surfaces*, Vol. I, to appear.

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